Variable Structure Controller Design for Robust Tracking and Model Following

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I. Introduction

A VARIABLE structure control system (VSCS)[1–3] is characterized by a control structure that is switched as the system state crosses a certain discontinuity surface in the state space. The central topic in VSCS is the so-called sliding mode. It is attained by designing the control strategy, which steers the state of the system into the sliding surface and thereafter maintains the motion of state trajectory within this subspace, such that the stability of the system is assured.

In the past few years, model-following control has become one of the most efficient and systematic procedures for the control of the linear time-invariant plant with unknown parameters or uncertainty. In this method the control objectives are specified in terms of a reference model and the design problem consists of determining the control input so that error between the plant and model approaches zero asymptotically.

The application of the VSCS theory to the model-following control systems was initiated by Young.[4] The advantage of designing a variable structure model-following control system is that the transient response of the model plant error, during sliding mode, can be prescribed in advance and is insensitive to the variation of the plant parameters and external disturbance. Thereafter, several approaches of the model-following control systems based on VSCS were presented.[5–9] However, in the previous approaches, the order of the reference model had to be equal to the plant order. Therefore, in the present work, an effective variable structure controller for model-following systems is proposed to release this limitation.

II. System Description and Problem Statement

Consider a class of uncertain/nonlinear dynamic systems modeled by

$$\dot{x}(t) = [A + \Delta A(x, t)]x(t) + Bu(t) + f(t, x, u)$$
$$y(t) = Cx(t) \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, and $y(t) \in R^p$; $\Delta A(x, t)$ represents the uncertainty portion of the plant and $f(t, x, u)$ symbolizes the disturbance to the plant and/or nonlinearities in the input. We give the following assumptions.

Assumption 1. There exist matrix functions $d(x, t), h(t, x, u)$ such that $\Delta A(x, t) = Bd(x, t)$ and $f(t, x, u) = Bh(t, x, u)$.

Assumption 2. There exist two known nonnegative scalars ρ_1 and ρ_2 such that $\|d(x, t)\| \le \rho_1$ and $\|h(t, x, u)\| \le \rho_2$, where $\| \cdot \|$ denotes the induced norm.

Assumption 3. The pair (A, B) is controllable, and the state vector $x(t)$ is accessible.

The reference model is described by

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t)$$
$$y_m(t) = C_m x_m(t) \tag{2}$$

where $x_m(t) \in R^k, r(t) \in R^d$, and $y_m(t) \in R^p$. Note that the order of the reference model k is allowed to be unequal to the plant order n. Our goal is to design a VSCS such that $y(t)$ tracks $y_m(t)$. To ensure that $y_m(t)$ can be tracked by $y(t)$, further assumptions are given.

Assumption 4. There exist matrices $F \in R^{n\times k}$ and $H \in R^{m\times k}$ such that

$$FA_m = AF + BH, \qquad C_m = CF \tag{3}$$

Assumption 5. There exists a matrix $D \in R^{m\times d}$ such that $FB_m = BD$.

Now, define the error between the model state and the plant state as

$$e(t) = x(t) - Fx_m(t) \tag{4}$$

Differentiating Eq. (4) with respect to time and using Eqs. (1–3), we obtain

$$\dot{e}(t) = Ae(t) + Bu(t) + \Delta A(x, t)x(t) + f(t, x, u)$$
$$-BHx_m(t) - FB_m r(t) \tag{5}$$

Remark 1. It can be seen that if $e(t) = 0$, then $y(t) = CFx_m(t) = C_m x_m(t) = y_m(t)$.

III. Variable Structure Controller Design

We begin our design by examining the dynamic behavior of Eq. (5) in the sliding mode:

$$S(t) = Ge(t) = G[x(t) - Fx_m(t)] = 0 \tag{6}$$

If $S(t) = 0, \forall t \ge t_s$, then $\dot{S}(t) = G\dot{e}(t) = 0$. Combining Eq. (6) with Eq. (5), we obtain the equivalent control u_{eq} of Eq. (5) in the sliding mode equation (6) as

$$u_{eq} = -(GB)^{-1}(GAe + G\Delta Ax + Gf - GBHx_m - GFB_m r) \tag{7}$$

Inserting u_{eq} into Eq. (5) and reshuffling terms yields

$$\dot{e} = [I - B(GB)^{-1}G](Ae + \Delta Ax + f - BHx_m - FB_m r) \tag{8}$$

Because assumptions 1 and 5 hold, the preceding equation can be reduced to[1]

$$\dot{e} = [I - B(GB)^{-1}G]Ae \tag{9}$$

After designing the sliding surfaces, we have to design a switching controller to establish the existence and reachability of a sliding mode.

Theorem 1. The reaching condition $S^T\dot{S} < 0$ of the sliding mode is satisfied if the control of error dynamics equation (5) is given by

$$u(t) = Hx_m + K_e e + K_m r + K_h + K_p \tag{10}$$

where $K_e = -(GB)^{-1}GA$, $K_m = (GB)^{-1}GFB_m$, $K_h = \alpha\rho_1(GB)^{-1}\|GB\|\|x\|\text{sgn}(S)$, $K_p = \eta\rho_2(GB)^{-1}\|GB\|\text{sgn}(S)$, $\alpha < -1, \eta < -1$, $\text{sgn}(S) = \text{sgn}(s_1)\,\text{sgn}(s_2)\cdots\text{sgn}(s_m)]^T$, and $\text{sgn}(s_i) = 1$ if $s_i > 0$, $\text{sgn}(s_i) = 0$ if $s_i = 0$, and $\text{sgn}(s_i) = -1$ if $s_i < 0$.

Proof. Combining Eq. (5) with Eq. (6), we obtain

$$\begin{aligned} S^T\dot{S} &= S^T[(GA + GBK_e)e + (GBK_m - GFB_m)r \\ &\quad + (GBK_h + G\Delta Ax) + (GBK_p + Gf)] \\ &= S^T[(GBK_h + G\Delta Ax) + (GBK_p + Gf)] \\ &\le [\alpha S^T\text{sgn}(S) + \|S\|]\|GB\|\|x\|\rho_1 \\ &\quad + [\eta S^T\text{sgn}(S) + \|S\|]\|GB\|\rho_2 \end{aligned}$$

From the definition of vector norms, we have $S^T\text{sgn}(S) \ge \|S\|$. If $\alpha, \eta < -1$, then $S^T\dot{S} < 0$. □

To implement the control, we need F and H. To solve it, we start by assuming

$$\text{rank}\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + p$$

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This condition is satisfied if the nominal system is controllable and the number of inputs is not less than the number of outputs. Rewrite Eq. (3) as

$$\begin{bmatrix} F \\ H \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^T \left(\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^T \right)^{-1} \begin{bmatrix} FA_m \\ C_m \end{bmatrix}$$

$$= \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \begin{bmatrix} FA_m \\ C_m \end{bmatrix} \tag{11}$$

where ψ_{11}, ψ_{12}, ψ_{21}, and ψ_{22} are of appropriate dimensions. An equivalent form of Eq. (11) is

$$F = \psi_{11} FA_m + \psi_{12} C_m \tag{12}$$

$$H = \psi_{21} FA_m + \psi_{22} C_m \tag{13}$$

Thus we can solve F from Eq. (12) and then solve H from Eq. (13). The method for solving F is summarized in the following.[10]

Define $\mathrm{Vec}(M)$ with an r-column matrix $M = [\boldsymbol{m}_1 \ \boldsymbol{m}_2 \ \cdots \ \boldsymbol{m}_r]$ as

$$\mathrm{Vec}(M) \equiv \begin{bmatrix} \boldsymbol{m}_1^T & \boldsymbol{m}_2^T & \cdots & \boldsymbol{m}_r^T \end{bmatrix}^T \tag{14}$$

and define the Kronecker product $\otimes$ for two matrices $P \in R^{m\times n}$ and $Q \in R^{r\times s}$ by

$$P \otimes Q = \begin{bmatrix} p_{11}Q & p_{12}Q & \cdots & p_{1n}Q \\ p_{21}Q & p_{21}Q & \cdots & p_{2n}Q \\ \vdots & & & \\ p_{m1}Q & p_{m2}Q & \cdots & p_{mn}Q \end{bmatrix} \tag{15}$$

Using Eqs. (14) and (15), Eq. (12) can be written in the vector form

$$\left(I_{nk\times nk} - A_m^T \otimes \psi_{11}\right)\mathrm{Vec}(F) = \mathrm{Vec}(\psi_{12} C_m) \tag{16}$$

Thus F and H can be solved from Eqs. (16) and (13). Once F and H have been derived, the variable-structure-model-following control can be designed according to Eq. (10).

Remark 2. If Eq. (16) has no solution for F, another A_m for the reference model should be chosen.

IV. Illustrative Example

Consider Advanced Fighter Technology Integration AFTI/F-16 aircraft flying at 3000 ft and Mach number 0.6 (Ref. 10). The system is described by Eq. (1) with $\boldsymbol{x} = [x_1 \ x_2 \ x_3]$ and $\boldsymbol{u} = [u_1 \ u_2]$ in which x_1, x_2, and x_3 are the pitch attitude, pitch rate, and angle of attack, respectively; u_1 and u_2 are the elevator deflection and flaperon deflection, respectively;

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.87 & 42.22 \\ 0 & 0.99 & -1.34 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ -17.25 & -1.58 \\ -0.17 & -0.25 \end{bmatrix}$$

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta_1 & \delta_2 \\ 0 & \delta_3 & \delta_4 \end{bmatrix} \qquad C = [1 \quad 0 \quad 0]$$

The uncertainty constraint set is given by $R = \{\delta\colon \|\delta_1\| \le 0.1, \|\delta_2\| \le 5, \|\delta_3\| \le 0.1, \|\delta_4\| \le 0.15\}$. The signal to be tracked is the output from the model given by

$$\dot{\boldsymbol{x}}_m = \begin{bmatrix} 0 & \pi/10 \\ -\pi/10 & 0 \end{bmatrix} \boldsymbol{x}_m, \qquad y_m = [1 \quad 0]\boldsymbol{x}_m$$

Following the algorithm, we get

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 0.3142 \\ 0.0226 & 0.1514 \end{bmatrix}$$

and

$$H = \begin{bmatrix} 0.0598 & 0.3480 \\ 0.0284 & 0.1677 \end{bmatrix}$$

Let

$$G = \begin{bmatrix} 10 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The uncertainty bound is $\|\Delta A\| \le \rho_1 = 5.0033$. Then, the variable structure controller of Eq. (10) can be given by

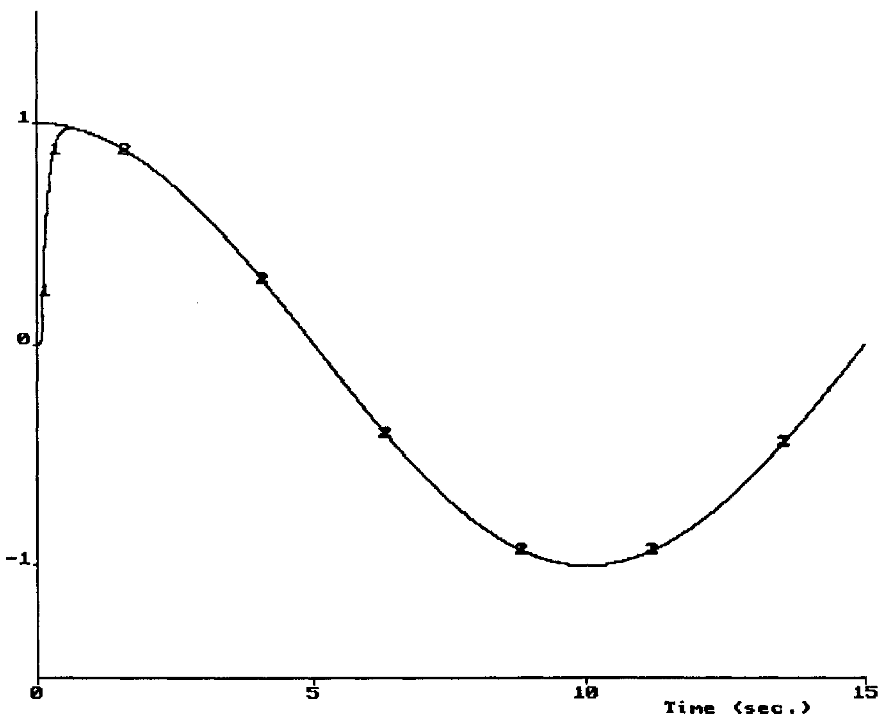

Fig. 1 Output responses of the system and model: 1:y, system output and 2:y_m, model output.

$$K_p = 0, \qquad K_e = \begin{bmatrix} 0 & 0.1776 & 3.1337 \\ 0 & 3.8392 & -7.4909 \end{bmatrix}, \qquad K_m = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$K_h = \begin{bmatrix} 8.0374 & -50.7996 \\ -5.4655 & 554.5829 \end{bmatrix} \begin{bmatrix} \mathrm{sgn}(s_1) \\ \mathrm{sgn}(s_2) \end{bmatrix} \|\boldsymbol{x}\|$$

In the simulations, we have $\alpha = -1.5$, $\delta_1(t) = 0.1\sin(t)$, $\delta_2(t) = 5\cos(t)$, $\delta_3(t) = 0.1\sin(2t)$, $\delta_4(t) = 0.15\cos(2t)$, $\boldsymbol{x}(0) = [0\ \ 0\ \ 0]^T$, and $\boldsymbol{x}_m(0) = [1\ \ 0]^T$. The output of the system is shown in Fig. 1. From simulation results, it can be seen that the proposed variable structure controller effectively makes the system output robustly track the reference model output.

V. Conclusions

A more general variable structure model-following design has been proposed in this work. It has been shown that the proposed method allows that the order of the reference model is not necessarily equal to the plant order. Moreover, the proposed variable-structure-model-following control retains the benefits of the sliding mode, which provides, in addition to the asymptotic stability, insensitivity to the parameter variations and disturbances to the error dynamics. The effectiveness of the proposed control method is demonstrated by a numerical example.

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